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COMPLETENESS VIA CORRESPONDENCE FOR EXTENSIONS OF THE LOGIC OF PARADOX

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Abstract. Taking our inspiration from modal correspondence theory, we present the idea of correspondence analysis for many-valued logics. As a benchmark case, we study truth-functional extensions of the Logic of Paradox (LP). First, we characterize each of the possible truth table entries for unary and binary operators that could be added to LP by an inference scheme. Second, we define a class of natural deduction systems on the basis of these characterizing inference schemes and a natural deduction system for LP. Third, we show that each of the resulting natural deduction systems is sound and complete with respect to its particular semantics.

§1. Introduction. The three-valued Logic of Paradox (LP) (Priest, 1979) is, unlike classical propositional logic, not functionally complete. Among other things, this means that adding unary truth-functional operators ($\sim$) or binary truth-functional operators ($\circ$) to LP's negation ($\neg$), disjunction ($\vee$), and conjunction ($\wedge$) poses special challenges for the construction of proof systems for such logics.[1] Given a logic $LP(\sim)_m(\circ)_n$ obtained by adding m truth-tables for unary operators $\sim_1, \ldots, \sim_m$ and n truth-tables for binary operators $\circ_1, \ldots, \circ_n$ to LP's truth-tables for $\neg$, $\vee$, $\wedge$ and LP's concept of validity, how are we to construct a proof system for it? We provide a uniform method that generates a natural deduction system for each logic $LP(\sim)_m(\circ)_n$.

To do so, we take the notion of *correspondence theory* from modal logic and adapt it to the study of many-valued logics such as LP. In modal logic, correspondence theory comprises model-theoretic and proof-theoretic concepts and methods that are based on structural relations between, on the one hand, first-order (and higher-order) formulas and, on the other, modal formulas and inference schemes. For example, for any Kripke frame $\mathfrak{F}$ it holds that the first-order formula $\forall x Rxx$ is true of $\mathfrak{F}$'s accessibility relation R if and only if the modal formula $\Box\phi \rightarrow \phi$ is true on $\mathfrak{F}$. The modal formula is then said to *characterize* the property expressed by the first-order formula. Moreover, adding the modal formula $\Box\phi \rightarrow \phi$ as an axiom to an axiom system for the basic modal logic K yields

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[1] Well-known three-valued logics that result from adding unary or binary truth-functional operators to LP are RM_3 (Anderson & Belnap, 1975) and J_3 (D'Ottaviano & da Costa, 1970; Epstein & D'Ottaviano, 2000).

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an axiom system which is sound and complete with respect to the class of all reflexive frames.[2]

In this paper, we show that something similar can be done for a many-valued logic such as LP. For example, for any truth table $f_\supset$ it holds that the first-order formula $\forall x \forall y(f_\supset(x, y) = 0 \rightarrow f_\supset(x, 0) = 0)$ is true of $f_\supset$ if and only if the inference scheme $\phi \supset (\phi \supset \psi)/\phi \supset \psi$ is valid according to $f_\supset$. The inference scheme is then said to *characterize* the property expressed by the first-order formula. We show that for every single entry E in a truth table f for a unary or a binary operator there is an inference scheme Π/ϕ such that E is an entry in f if and only if Π/ϕ is valid according to f. As a consequence, each truth table for a unary (or binary) operator can be characterized in terms of three (or nine) inference schemes. Moreover, adding the inference schemes that characterize a truth table f as derivation rules to a natural deduction system for LP yields a natural deduction system which is sound and complete with respect to the semantics that also contains, next to LP's truth-tables for $\neg$, $\vee$, and $\wedge$, the truth table f. In this way, we obtain a natural deduction system for each logic $LP(\sim)_m(\circ)_n$.

The structure of our paper is as follows. First, we present a correspondence analysis for LP and characterize each of the 9 possible entries in the truth table for a unary operator $\sim$ and each of the 27 possible entries in a truth table for a binary operator $\circ$ by an inference scheme. Second, we define a class of natural deduction systems on the basis of a natural deduction system for LP and the 9 plus 27 characterizing inference schemes. Third, we show that each of the resulting natural deduction systems is sound and complete with respect to its particular semantics.

§2. Correspondence analysis for LP. The three-valued logic LP evaluates arguments consisting of formulas from a propositional language $\mathcal{L}$ built from a set $\mathcal{P} = \{p, p', \dots\}$ of atomic formulas using negation ($\neg$), disjunction ($\vee$), and conjunction ($\wedge$). In LP, a valuation is a function v from the set $\mathcal{P}$ of atomic formulas to the set $\{0, i, 1\}$ of truth-values 'false', 'both', and 'true'. A valuation v on $\mathcal{P}$ is extended to a valuation on $\mathcal{L}$ according to the truth-tables for $\neg$, $\vee$, and $\wedge$:

$f_\neg$	
0	1
i	i
1	0

$f_\vee$	0	i	1
0	0	i	1
i	i	i	1
1	1	1	1

$f_\wedge$	0	i	1
0	0	0	0
i	0	i	i
1	0	i	1

An inference scheme from a set Π of premises to a conclusion ϕ is *valid* (notation: $\Pi \models \phi$) if and only if for each valuation v it holds that if $v(\psi) \neq 0$ for all ψ in Π, then $v(\phi) \neq 0$.

Let $\mathcal{L}_{(\sim)_m(\circ)_n}$ be the language built from the set $\mathcal{P} = \{p, p', \dots\}$ of atomic formulas using $\neg$, $\vee$, $\wedge$, unary operators $\sim_1, \dots, \sim_m$, and binary operators $\circ_1, \dots, \circ_n$. Clearly, $\mathcal{L}_{(\sim)_m(\circ)_n}$ is an extension of $\mathcal{L}$. To interpret this extended language, we use, next to LP's truth-tables $f_\neg$, $f_\vee$, and $f_\wedge$, the truth-tables $f_{\sim_1}, \dots, f_{\sim_m}$ and $f_{\circ_1}, \dots, f_{\circ_n}$. We refer to the resulting logic as $LP(\sim)_m(\circ)_n$. Which inference schemes are (in)valid in the logic $LP(\sim)_m(\circ)_n$ ultimately depends on the entries in the truth-tables of its operators. To study these dependencies in a precise way, we present a single entry correspondence analysis for LP.

[2] The first studies in correspondence theory for modal logics were by Sahlqvist (1975) and van Benthem (1976). For an up-to-date review of modal correspondence theory, see van Benthem (2001).

DEFINITION 2.1 (Correspondence) *Let $\Pi \cup \{\phi\} \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$. Let $x, y, z \in \{0, i, 1\}$. Let E be a truth table entry of the type $f_\sim(x) = y$ or $f_\circ(x, y) = z$. Then the truth table entry E is characterized by an inference scheme Π/ϕ, if*

$$E \text{ if and only if } \Pi \models \phi.$$

Accordingly, each of the 9 possible entries in a truth table $f_\sim$ and each of the 27 possible entries in a truth table $f_\circ$ is characterized by an inference scheme (we do the binary case first):

THEOREM 2.2. *Let $\phi, \psi, \chi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. Then*

$$f_\circ(0,0)=\begin{cases} 0 & \textit{iff} \quad \phi \circ \psi \models \phi \vee \psi \\ i & \textit{iff} \quad \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee (\phi \vee \psi) \\ 1 & \textit{iff} \quad \neg(\phi \circ \psi) \models \phi \vee \psi \end{cases}$$

$$f_\circ(0,i)=\begin{cases} 0 & \textit{iff} \quad \psi \wedge \neg\psi, \phi \circ \psi \models \phi \\ i & \textit{iff} \quad \psi \wedge \neg\psi \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee \phi \\ 1 & \textit{iff} \quad \psi \wedge \neg\psi, \neg(\phi \circ \psi) \models \phi \end{cases}$$

$$f_\circ(0,1)=\begin{cases} 0 & \textit{iff} \quad \phi \circ \psi \models \phi \vee \neg\psi \\ i & \textit{iff} \quad \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee (\phi \vee \neg\psi) \\ 1 & \textit{iff} \quad \neg(\phi \circ \psi) \models \phi \vee \neg\psi \end{cases}$$

$$f_\circ(i,0)=\begin{cases} 0 & \textit{iff} \quad \phi \wedge \neg\phi, \phi \circ \psi \models \psi \\ i & \textit{iff} \quad \phi \wedge \neg\phi \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee \psi \\ 1 & \textit{iff} \quad \phi \wedge \neg\phi, \neg(\phi \circ \psi) \models \psi \end{cases}$$

$$f_\circ(i,i)=\begin{cases} 0 & \textit{iff} \quad \phi \wedge \neg\phi, \psi \wedge \neg\psi, \phi \circ \psi \models \chi \\ i & \textit{iff} \quad \phi \wedge \neg\phi, \psi \wedge \neg\psi \models (\phi \circ \psi) \wedge \neg(\phi \circ \psi) \\ 1 & \textit{iff} \quad \phi \wedge \neg\phi, \psi \wedge \neg\psi, \neg(\phi \circ \psi) \models \chi \end{cases}$$

$$f_\circ(i,1)=\begin{cases} 0 & \textit{iff} \quad \phi \wedge \neg\phi, \phi \circ \psi \models \neg\psi \\ i & \textit{iff} \quad \phi \wedge \neg\phi \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee \neg\psi \\ 1 & \textit{iff} \quad \phi \wedge \neg\phi, \neg(\phi \circ \psi) \models \neg\psi \end{cases}$$

$$f_\circ(1,0)=\begin{cases} 0 & \textit{iff} \quad \phi \circ \psi \models \neg\phi \vee \psi \\ i & \textit{iff} \quad \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee (\neg\phi \vee \psi) \\ 1 & \textit{iff} \quad \neg(\phi \circ \psi) \models \neg\phi \vee \psi \end{cases}$$

$$f_\circ(1,i)=\begin{cases} 0 & \textit{iff} \quad \psi \wedge \neg\psi, \phi \circ \psi \models \neg\phi \\ i & \textit{iff} \quad \psi \wedge \neg\psi \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee \neg\phi \\ 1 & \textit{iff} \quad \psi \wedge \neg\psi, \neg(\phi \circ \psi) \models \neg\phi \end{cases}$$

$$f_\circ(1,1)=\begin{cases} 0 & \textit{iff} \quad \phi \circ \psi \models \neg\phi \vee \neg\psi \\ i & \textit{iff} \quad \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee (\neg\phi \vee \neg\psi) \\ 1 & \textit{iff} \quad \neg(\phi \circ \psi) \models \neg\phi \vee \neg\psi. \end{cases}$$

Proof. We prove the cases for $f_\circ(0,0) = 0$, $f_\circ(i,i) = 1$, and $f_\circ(1,i) = i$.

Case $f_\circ(0,0) = 0$. ($\Rightarrow$) Suppose that $\phi \circ \psi \not\models \phi \vee \psi$. Then there is a valuation v such that $v(\phi \circ \psi) \neq 0$ and $v(\phi \vee \psi) = 0$. Then $v(\phi) = 0$, $v(\psi) = 0$, and $v(\phi \circ \psi) \neq 0$. Therefore, it must be that $f_\circ(0,0) \neq 0$. ($\Leftarrow$) Suppose that $\phi \circ \psi \models \phi \vee \psi$. Then $p \circ q \models p \vee q$, where p and q are atomic formulas. Then for every valuation v it holds that if $v(p \circ q) \neq 0$, then $v(p \vee q) \neq 0$. Then for every valuation it holds that if $v(p) = 0$ and $v(q) = 0$, then $v(p \circ q) = 0$. Therefore, it must be that $f_\circ(0,0) = 0$.

Case $f_\circ(i,i) = 1$. ($\Rightarrow$) Suppose that $\phi \wedge \neg\phi, \psi \wedge \neg\psi, \neg(\phi \circ \psi) \not\models \chi$. Then there is a valuation v such that $v(\phi \wedge \neg\phi) \neq 0$, $v(\psi \wedge \neg\psi) \neq 0$, $v(\neg(\phi \circ \psi)) \neq 0$, and $v(\chi) = 0$. Then $v(\phi) = i$, $v(\psi) = i$, and $v(\phi \circ \psi) \neq 1$. Therefore, it must be that $f_\circ(i,i) \neq 1$. ($\Leftarrow$) Suppose that $\phi \wedge \neg\phi, \psi \wedge \neg\psi, \neg(\phi \circ \psi) \models \chi$. Then $p \wedge \neg p, q \wedge \neg q, \neg(p \circ q) \models r$, where p, q, and r are atomic formulas. Then for every valuation v it holds that if $v(p \wedge \neg p) \neq 0$, $v(q \wedge \neg q) \neq 0$, and $v(\neg(p \circ q)) \neq 0$, then $v(r) \neq 0$. Then, since $v(r)$ is independent from $v(p)$ and $v(q)$, it must be that for every valuation v it holds that if $v(p) = i$ and $v(q) = i$, then $v(p \circ q) = 1$. Therefore, it must be that $f_\circ(i,i) = 1$.

Case $f_\circ(1,i) = i$. ($\Rightarrow$) Suppose that $\psi \wedge \neg\psi \not\models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee \neg\phi$. Then there is a valuation v such that $v(\psi \wedge \neg\psi) \neq 0$ and $v(((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee \neg\phi) = 0$. Then $v(\phi) = 1$, $v(\psi) = i$, and $v(\phi \circ \psi) \neq i$. Therefore, it must be that $f_\circ(1,i) \neq i$. ($\Leftarrow$) Suppose that $\psi \wedge \neg\psi \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee \neg\phi$. Then $q \wedge \neg q \models ((p \circ q) \wedge \neg(p \circ q)) \vee \neg p$, where p and q are atomic formulas. Then for every valuation v it holds that if $v(q \wedge \neg q) \neq 0$, then $v(((p \circ q) \wedge \neg(p \circ q)) \vee \neg p) \neq 0$. Then for every valuation v it holds that if $v(q) = i$, then $v(p \circ q) = i$ or $v(p) \neq 1$. Then for every valuation v it holds that if $v(p) = 1$ and $v(q) = i$, then $v(p \circ q) = i$. Therefore, it must be that $f_\circ(1,i) = i$.

The other cases are proved similarly. □

COROLLARY 2.3. *Let $\phi, \psi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. Then*

$$f_\sim(0) = \begin{cases} 0 & \textit{iff} \quad \sim\phi \models \phi \\ i & \textit{iff} \quad \models (\sim\phi \wedge \neg\sim\phi) \vee \phi \\ 1 & \textit{iff} \quad \neg\sim\phi \models \phi \end{cases}$$

$$f_\sim(i) = \begin{cases} 0 & \textit{iff} \quad \phi \wedge \neg\phi, \sim\phi \models \psi \\ i & \textit{iff} \quad \phi \wedge \neg\phi \models \sim\phi \wedge \neg\sim\phi \\ 1 & \textit{iff} \quad \phi \wedge \neg\phi, \neg\sim\phi \models \psi \end{cases}$$

$$f_\sim(1) = \begin{cases} 0 & \textit{iff} \quad \sim\phi \models \neg\phi \\ i & \textit{iff} \quad \models (\sim\phi \wedge \neg\sim\phi) \vee \neg\phi \\ 1 & \textit{iff} \quad \neg\sim\phi \models \neg\phi. \end{cases}$$

Proof. Adapt the cases $f_\circ(0,0)$, $f_\circ(i,i)$, and $f_\circ(1,1)$. □

As a consequence, given LP's truth-tables $f_\neg$, $f_\vee$, $f_\wedge$, and its concept of validity, each unary operator $\sim_k$ $(1 \leq k \leq m)$ in the logic $LP(\sim)_m(\circ)_n$ is characterized by the set of three inference schemes that characterize the three entries in its truth table $f_{\sim_k}$. Likewise, each binary operator $\circ_l$ $(1 \leq l \leq n)$ in the logic $LP(\sim)_m(\circ)_n$ is characterized by the set of nine inference schemes that characterize the nine entries in its truth table $f_{\circ_l}$. Note that the inference schemes that characterize a truth table are independent.

§3. Natural deduction systems. We show that for each logic $LP(\sim)_m(\circ)_n$ it holds that if we add the three characterizing inference schemes of each unary operator

$\sim_k$ $(1 \leq k \leq m)$ and the nine characterizing inference schemes of each binary operator $\circ_l$ $(1 \leq l \leq n)$ as derivation rules to a natural deduction system for LP, we obtain a sound and complete proof system for it.

The proof system $\mathbf{ND}_{LP}$ is defined as follows.[3] It is a corollary of our main theorem that $\mathbf{ND}_{LP}$ is sound and complete with respect to LP.

DEFINITION 3.1 *Derivations in the system* $\mathbf{ND}_{LP}$ *are inductively defined as follows:*

Basis: The proof tree with a single occurrence of an assumption ϕ *is a derivation.*

Induction Step: Let $\mathcal{D}$, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ *be derivations. Then they can be extended by the following rules (double lines indicate that the rules work both ways):*

$$\begin{array}{c} \\ \hline \phi \vee \neg\phi \end{array} EM$$

$$\begin{array}{c} \mathcal{D}_1 \qquad \mathcal{D}_2 \\ \phi \qquad\quad \psi \\ \hline \phi \wedge \psi \end{array} \wedge I \qquad \begin{array}{c} \mathcal{D} \\ \phi \wedge \psi \\ \hline \phi \end{array} \wedge E_1 \qquad \begin{array}{c} \mathcal{D} \\ \phi \wedge \psi \\ \hline \psi \end{array} \wedge E_2$$

$$\begin{array}{c} \mathcal{D} \\ \phi \\ \hline \phi \vee \psi \end{array} \vee I_1 \qquad \begin{array}{c} \mathcal{D} \\ \psi \\ \hline \phi \vee \psi \end{array} \vee I_2 \qquad \begin{array}{ccc} & [\phi]^u & [\psi]^v \\ \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 \\ \phi \vee \psi & \chi & \chi \\ \hline & \chi & \end{array} \vee E^{u,v}$$

$$\begin{array}{c} \mathcal{D} \\ \phi \\ \hline\hline \neg\neg\phi \end{array} DN \qquad \begin{array}{c} \mathcal{D} \\ \neg(\phi \vee \psi) \\ \hline\hline \neg\phi \wedge \neg\psi \end{array} DeM_{\vee} \qquad \begin{array}{c} \mathcal{D} \\ \neg(\phi \wedge \psi) \\ \hline\hline \neg\phi \vee \neg\psi \end{array} DeM_{\wedge}$$

Theorem 2.2 and Corollary 2.3 tell us that each truth table $f_{\sim}$ is characterized by three inference schemes and that each truth table $f_{\circ}$ is characterized by nine inference schemes. What we add to the proof system $\mathbf{ND}_{LP}$ are these characterizing inference schemes turned into derivation rules. To be precise, for each inference scheme $\psi_1, \ldots, \psi_j/\phi$ that characterizes an entry $f_{\sim}(x) = y$ in the truth table $f_{\sim}$, we add the rule

$$\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_j \\ \psi_1 & \cdots & \psi_j \\ \hline & \phi & \end{array} R_{\sim}(x, y)$$

to the system $\mathbf{ND}_{LP}$. Likewise, for each inference scheme $\psi_1, \ldots, \psi_j/\phi$ that characterizes an entry $f_{\circ}(x, y) = z$ in the truth table $f_{\circ}$, we add the rule

$$\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_j \\ \psi_1 & \cdots & \psi_j \\ \hline & \phi & \end{array} R_{\circ}(x, y, z)$$

to the system $\mathbf{ND}_{LP}$.

For example, suppose $f_{\circ}(0, 0) = 0$ is one of the truth table entries in $f_{\circ}$. Then, because $\phi \circ \psi/\phi \vee \psi$ characterizes $f_{\circ}(0, 0) = 0$, we add the rule

$$\begin{array}{c} \mathcal{D} \\ \phi \circ \psi \\ \hline \phi \vee \psi \end{array} R_{\circ}(0, 0, 0)$$

to our proof system.

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[3] The notational conventions are given in Troelstra & Schwichtenberg (1996).

In this way, we obtain a natural deduction system $\mathbf{ND}_{LP} + \bigcup_{k=1}^{m}\{R_{\sim_k}(x, y) : f_{\sim_k}(x) = y\} + \bigcup_{l=1}^{n}\{R_{\circ_l}(x, y, z) : f_{\circ_l}(x, y) = z\}$, which we refer to as $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. Thus, any combination of a choice of m (out of $3^3 = 27$) truth-tables for unary operators and a choice of n (out of $3^9 = 19683$) truth-tables for binary operators defines a natural deduction system $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. Hence, we get $2^{27} \times 2^{19683}$ natural deduction systems. We prove their soundness and completeness in one go.

3.1. Soundness of $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. A conclusion ϕ is *derivable* from a set Π of premises (notation: $\Pi \vdash \phi$) if and only if there is a derivation in the system $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$ of ϕ from Π.

LEMMA 3.2 (Local Soundness). *Let* $\Pi, \Pi', \Pi'' \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$ *and let* $\phi, \psi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. *Then*

(i) *If* $\phi \in \Pi$, *then* $\Pi \models \phi$
(ii) $\Pi \models \phi \vee \neg\phi$
(iii) *If* $\Pi \models \phi$ *and* $\Pi' \models \psi$, *then* $\Pi, \Pi' \models \phi \wedge \psi$
(iv) *If* $\Pi \models \phi \wedge \psi$, *then* $\Pi \models \phi$
(v) *If* $\Pi \models \phi \wedge \psi$, *then* $\Pi \models \psi$
(vi) *If* $\Pi \models \phi$, *then* $\Pi \models \phi \vee \psi$
(vii) *If* $\Pi \models \psi$, *then* $\Pi \models \phi \vee \psi$
(ix) *If* $\Pi \models \phi \vee \psi$ *and* $\Pi', \phi \models \chi$ *and* $\Pi'', \psi \models \chi$, *then* $\Pi, \Pi', \Pi'' \models \chi$
(x) $\Pi \models \phi$ *if and only if* $\Pi \models \neg\neg\phi$
(xi) $\Pi \models \neg(\phi \vee \psi)$ *if and only if* $\Pi \models \neg\phi \wedge \neg\psi$
(xii) $\Pi \models \neg(\phi \wedge \psi)$ *if and only if* $\Pi \models \neg\phi \vee \neg\psi$.

THEOREM 3.3 (Soundness). *Let* $\Pi \cup \{\phi\} \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$. *Then*

$$\text{If } \Pi \vdash \phi, \text{ then } \Pi \models \phi.$$

Proof. The proof is by induction on the depth of derivation. The local soundness of the rules of $\mathbf{ND}_{LP}$ follows from Lemma 3.2. For each unary operator $\sim_k$ ($1 \leq k \leq m$), the local soundness of the three rules in $\{R_{\sim_k}(x, y) : f_{\sim_k}(x) = y\}$ follows from Corollary 2.3. For each binary operator $\circ_l$ ($1 \leq l \leq n$), the local soundness of the nine rules in $\{R_{\circ_l}(x, y, z) : f_{\circ_l}(x, y) = z\}$ follows from Theorem 2.2. □

3.2. Completeness of $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. In our completeness proof, nontrivial prime theories are the syntactical counterparts of valuations. Any set of formulas that is (i) not equal to the whole language, (ii) closed under derivability, and (iii) closed under the disjunction property is a *nontrivial prime theory* (NPT):

DEFINITION 3.4 *Let* $\Pi \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$. *Then* Π *is a* nontrivial prime theory *(NPT), if it is*

nontrivial: $\Pi \neq \mathcal{L}_{(\sim)_m(\circ)_n}$
closed: *If* $\Pi \vdash \phi$, *then* $\phi \in \Pi$
prime: *If* $\phi \vee \psi \in \Pi$, *then* $\phi \in \Pi$ *or* $\psi \in \Pi$.

DEFINITION 3.5 *Let* $\Pi \cup \{\phi\} \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$. *We define* ϕ*'s* elementhood in Π *(notation:* $e(\phi, \Pi)$*) as follows:*

$$e(\phi, \Pi) = \begin{cases} \emptyset, & \textit{if } \phi \notin \Pi \textit{ and } \neg\phi \notin \Pi \\ 0, & \textit{if } \phi \notin \Pi \textit{ and } \neg\phi \in \Pi \\ i, & \textit{if } \phi \in \Pi \textit{ and } \neg\phi \in \Pi \\ 1, & \textit{if } \phi \in \Pi \textit{ and } \neg\phi \notin \Pi. \end{cases}$$

LEMMA 3.6. *Let* Π *be an NPT and let* $\phi, \psi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. *Then*

(i) $e(\phi, \Pi) \neq \emptyset$

(ii) $f_\neg(e(\phi, \Pi)) = e(\neg\phi, \Pi)$

(iii) $f_\vee(e(\phi, \Pi), e(\psi, \Pi)) = e(\phi \vee \psi, \Pi)$

(iv) $f_\wedge(e(\phi, \Pi), e(\psi, \Pi)) = e(\phi \wedge \psi, \Pi)$

(v) $f_{\sim_k}(e(\phi, \Pi)) = e(\sim_k \phi, \Pi)$ *for* $1 \leq k \leq m$

(vi) $f_{\circ_l}(e(\phi, \Pi), e(\psi, \Pi)) = e(\phi \circ_l \psi, \Pi)$ *for* $1 \leq l \leq n$.

Proof.

(i) By the rule *EM*, it must be that $\Pi \vdash \phi \vee \neg\phi$. By closure and primeness, $\phi \in \Pi$ or $\neg\phi \in \Pi$. Therefore, $e(\phi, \Pi) \neq \emptyset$.

(ii) Suppose $e(\phi, \Pi) = 0$. Then $\phi \notin \Pi$ and $\neg\phi \in \Pi$. By closure and the rule *DN*, $\neg\phi \in \Pi$ and $\neg\neg\phi \notin \Pi$. Hence, $e(\neg\phi, \Pi) = 1 = f_\neg(0) = f_\neg(e(\phi, \Pi))$.
Suppose $e(\phi, \Pi) = i$. Then $\phi \in \Pi$ and $\neg\phi \in \Pi$. By closure and the rule *DN*, $\neg\phi \in \Pi$ and $\neg\neg\phi \in \Pi$. Hence, $e(\neg\phi, \Pi) = i = f_\neg(i) = f_\neg(e(\phi, \Pi))$.
Suppose $e(\phi, \Pi) = 1$. Then $\phi \in \Pi$ and $\neg\phi \notin \Pi$. By closure and the rule *DN*, $\neg\phi \notin \Pi$ and $\neg\neg\phi \in \Pi$. Hence, $e(\neg\phi, \Pi) = 0 = f_\neg(1) = f_\neg(e(\phi, \Pi))$.

(iii) We prove the cases for (1) $e(\phi, \Pi) = 0$ and $e(\psi, \Pi) = 0$, (2) $e(\phi, \Pi) = i$ and $e(\psi, \Pi) = i$, and (3) $e(\phi, \Pi) = 1$ and $e(\psi, \Pi) = i$. The other six cases are proved similarly.

  (1) Suppose $e(\phi, \Pi) = 0$ and $e(\psi, \Pi) = 0$. Then $\phi \notin \Pi$, $\psi \notin \Pi$, $\neg\phi \in \Pi$, and $\neg\psi \in \Pi$. By primeness, $\phi \vee \psi \notin \Pi$. By closure and the rules $\wedge I$ and $DeM_\vee$, $\neg(\phi \vee \psi) \in \Pi$. Hence, $e(\phi \vee \psi, \Pi) = 0 = f_\vee(0, 0) = f_\vee(e(\phi, \Pi), e(\psi, \Pi))$.

  (2) Suppose $e(\phi, \Pi) = i$ and $e(\psi, \Pi) = i$. Then $\phi \in \Pi$, $\psi \in \Pi$, $\neg\phi \in \Pi$, and $\neg\psi \in \Pi$. By closure and the rule $\vee I_1$, $\phi \vee \psi \in \Pi$. By closure and the rules $\wedge I$ and $DeM_\vee$, $\neg(\phi \vee \psi) \in \Pi$. Hence, $e(\phi \vee \psi, \Pi) = i = f_\vee(i, i) = f_\vee(e(\phi, \Pi), e(\psi, \Pi))$.

  (3) Suppose $e(\phi, \Pi) = 1$ and $e(\psi, \Pi) = i$. Then $\phi \in \Pi$, $\psi \in \Pi$, $\neg\phi \notin \Pi$, and $\neg\psi \in \Pi$. By closure and the rule $\vee I_1$, $\phi \vee \psi \in \Pi$. By closure and the rules $\wedge E_1$ and $DeM_\vee$, $\neg(\phi \vee \psi) \notin \Pi$. Hence, $e(\phi \vee \psi, \Pi) = 1 = f_\vee(1, i) = f_\vee(e(\phi, \Pi), e(\psi, \Pi))$.

(iv) Analogous to (iii).

(v) Analogous to (vi).

(vi) There are nine cases for each $\circ_l$ ($1 \leq l \leq n$). (For readability, we drop the subscript l in the remainder of this proof.) We prove the cases for (1) $e(\phi, \Pi) = 0$ and $e(\psi, \Pi) = 0$, (2) $e(\phi, \Pi) = i$ and $e(\psi, \Pi) = i$, and (3) $e(\phi, \Pi) = 1$ and $e(\psi, \Pi) = i$. The other six cases are proved similarly.

(1) Suppose $e(\phi, \Pi) = 0$ and $e(\psi, \Pi) = 0$. Then $\phi \notin \Pi$, $\psi \notin \Pi$, $\neg\phi \in \Pi$, and $\neg\psi \in \Pi$. There are three cases: (a), (b), and (c).

(a) Suppose $R_\circ(0, 0, 0)$ is one of the nine rules for $\circ$ in $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. Then $f_\circ(0, 0) = 0$. Suppose $\phi \circ \psi \in \Pi$. By closure, primeness, and the rule $R_\circ(0, 0, 0)$, it must be that $\phi \in \Pi$ or $\psi \in \Pi$. Contradiction. Hence, $\phi \circ \psi \notin \Pi$ and, by (i), $\neg(\phi \circ \psi) \in \Pi$. Therefore, $e(\phi \circ \psi, \Pi) = 0 = f_\circ(0, 0) = f_\circ(e(\phi, \Pi), e(\psi, \Pi))$.

(b) Suppose $R_\circ(0, 0, i)$ is one of the nine rules for $\circ$ in $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. Then $f_\circ(0, 0) = i$. By closure, primeness, and the rules $R_\circ(0, 0, i)$, $\wedge E_1$, and $\wedge E_2$, it must be that $\phi \circ \psi \in \Pi$ and $\neg(\phi \circ \psi) \in \Pi$. Therefore, $e(\phi \circ \psi, \Pi) = i = f_\circ(0, 0) = f_\circ(e(\phi, \Pi), e(\psi, \Pi))$.

(c) Suppose $R_\circ(0, 0, 1)$ is one of the nine rules for $\circ$ in $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. Analogous to (1)(a).

(2) Suppose $e(\phi, \Pi) = i$ and $e(\psi, \Pi) = i$. Then $\phi \in \Pi$, $\psi \in \Pi$, $\neg\phi \in \Pi$, and $\neg\psi \in \Pi$. There are three cases: (a), (b), and (c).

(a) Suppose $R_\circ(i, i, 0)$ is one of the nine rules for $\circ$ in $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. Then $f_\circ(i, i) = 0$. Suppose $\phi \circ \psi \in \Pi$. By closure and the rules $\wedge I$ and $R_\circ(i, i, 0)$, it must be that $\Pi = \mathcal{L}_{(\sim)_m(\circ)_n}$. But Π is nontrivial. Contradiction. Hence, $\phi \circ \psi \notin \Pi$ and, by (i), $\neg(\phi \circ \psi) \in \Pi$. Therefore, $e(\phi \circ \psi, \Pi) = 0 = f_\circ(i, i) = f_\circ(e(\phi, \Pi), e(\psi, \Pi))$.

(b) Suppose $R_\circ(i, i, i)$ is one of the nine rules for $\circ$ in $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. Then $f_\circ(i, i) = i$. By closure and the rules $\wedge I$, $\wedge E_1$, $\wedge E_2$, and $R_\circ(i, i, i)$, it must be that $\phi \circ \psi \in \Pi$ and $\neg(\phi \circ \psi) \in \Pi$. Therefore, $e(\phi \circ \psi, \Pi) = i = f_\circ(i, i) = f_\circ(e(\phi, \Pi), e(\psi, \Pi))$.

(c) Suppose $R_\circ(i, i, 1)$ is one of the nine rules for $\circ$ in $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. Analogous to (2)(a).

(3) Suppose $e(\phi, \Pi) = 1$ and $e(\psi, \Pi) = i$. Then $\phi \in \Pi$, $\psi \in \Pi$, $\neg\phi \notin \Pi$, and $\neg\psi \in \Pi$. There are three cases: (a), (b), and (c).

(a) Suppose $R_\circ(1, i, 0)$ is one of the nine rules for $\circ$ in $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. Then $f_\circ(1, i) = 0$. Suppose $\phi \circ \psi \in \Pi$. By closure and the rules $\wedge I$ and $R_\circ(1, i, 0)$, it must be that $\neg\phi \in \Pi$. Contradiction. Hence, $\phi \circ \psi \notin \Pi$ and, by (i), $\neg(\phi \circ \psi) \in \Pi$. Therefore, $e(\phi \circ \psi, \Pi) = 0 = f_\circ(1, i) = f_\circ(e(\phi, \Pi), e(\psi, \Pi))$.

(b) Suppose $R_\circ(1, i, i)$ is one of the nine rules for $\circ$ in $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. Then $f_\circ(1, i) = i$. By closure, primeness, and the rules $\wedge I$, $\wedge E_1$, $\wedge E_2$, and $R_\circ(1, i, i)$, it must be that $\phi \circ \psi \in \Pi$ and $\neg(\phi \circ \psi) \in \Pi$. Therefore, $e(\phi \circ \psi, \Pi) = i = f_\circ(1, i) = f_\circ(e(\phi, \Pi), e(\psi, \Pi))$.

(c) Suppose $R_\circ(1, i, 1)$ is one of the nine rules for $\circ$ in $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. Analogous to (3)(a). □

LEMMA 3.7 (Truth). *Let Π be an NPT. Let v_Π be the function that assigns to each atomic formula p in $\mathcal{P}$ the elementhood of p in Π: $v_\Pi(p) = e(p, \Pi)$ for all p in $\mathcal{P}$. Then for all ϕ in $\mathcal{L}_{(\sim)_m(\circ)_n}$ it holds that*

$$v_\Pi(\phi) = e(\phi, \Pi).$$

Proof. By a straightforward induction on ϕ. Use Lemma 3.6. □

LEMMA 3.8 (Lindenbaum). *Let* $\Pi \cup \{\phi\} \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$. *Suppose that* $\Pi \nvdash \phi$. *Then there is a set* $\Pi^* \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$ *such that*

(i) $\Pi \subseteq \Pi^*$

(ii) $\Pi^* \nvdash \phi$

(iii) Π^* *is an NPT.*

Proof. Suppose that $\Pi \nvdash \phi$. Let $\psi_1, \psi_2, \ldots$ be an enumeration of $\mathcal{L}_{(\sim)_m(\circ)_n}$. We define the sequence $\Pi_0, \Pi_1, \ldots$ of sets of formulas as follows:

$$\begin{aligned} \Pi_0 &= \Pi \\ \Pi_{i+1} &= \begin{cases} \Pi_i \cup \{\psi_{i+1}\}, & \text{if } \Pi_i \cup \{\psi_{i+1}\} \nvdash \phi \\ \Pi_i, & \text{otherwise.} \end{cases} \end{aligned}$$

Take $\Pi^* = \bigcup_{n \in \mathbb{N}} \Pi_n$. Then the claims (i), (ii), and (iii) hold:

(i) Obviously $\Pi \subseteq \Pi^*$.

(ii) Suppose $\Pi^* \vdash \phi$. Then there is a finite Π' such that $\Pi' \subseteq \Pi^*$ and $\Pi' \vdash \phi$, because derivations are finite. Then there must be an n in $\mathbb{N}$ such that $\Pi' \subseteq \Pi_n$. Then $\Pi_n \vdash \phi$. By the construction, $\Pi_n \nvdash \phi$. Contradiction. Therefore, $\Pi^* \nvdash \phi$.

(iii) To show that Π^* is an NPT, we have to show that (a) Π^* is closed, (b) Π^* is prime, and (c) Π^* is nontrivial.

(a) Suppose $\Pi^* \vdash \psi$. Then $\psi = \psi_n$ for some $n \in \mathbb{N}$. Suppose $\psi_n \notin \Pi_n$. By the construction, $\Pi_{n-1} \cup \{\psi_n\} \vdash \phi$. Because $\Pi_{n-1} \subseteq \Pi^*$ and $\Pi^* \vdash \psi_n$, it must be that $\Pi^* \vdash \phi$. This contradicts what was proved in (ii) of this lemma. Hence, $\psi \in \Pi^*$. Therefore, Π^* is closed.

(b) Suppose that $\psi \vee \chi \in \Pi^*$. Suppose $\psi \notin \Pi^*$ and $\chi \notin \Pi^*$. Then $\psi = \psi_m$ for some $m \in \mathbb{N}$ and $\chi = \psi_n$ for some $n \in \mathbb{N}$. By the construction, $\Pi_{m-1} \cup \{\psi_m\} \vdash \phi$ and $\Pi_{n-1} \cup \{\psi_n\} \vdash \phi$. Obviously, $\Pi^* \vdash \psi_m \vee \psi_n$. Note that $\Pi_m \subseteq \Pi^*$ and $\Pi_n \subseteq \Pi^*$. Hence, $\Pi^* \cup \{\psi_m\} \vdash \phi$ and $\Pi^* \cup \{\psi_n\} \vdash \phi$. By the rule $\vee E^{u,v}$, it must be that $\Pi^* \vdash \phi$. This contradicts what was proved in (ii) of this lemma. Hence, $\psi \in \Pi^*$ or $\chi \in \Pi^*$. Therefore, Π^* is prime.

(c) Because of what was proved in (ii) of this lemma, it must be that $\phi \notin \Pi^*$. Therefore, Π^* is nontrivial. □

THEOREM 3.9 (Completeness). *Let* $\Pi \cup \{\phi\} \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$. *Then*

$$\textit{If } \Pi \models \phi, \textit{ then } \Pi \vdash \phi.$$

Proof. By contraposition. Suppose $\Pi \nvdash \phi$. By Lemma 3.8, there is an NPT Π^* such that $\Pi \subseteq \Pi^*$ and $\Pi^* \nvdash \phi$. Let v_{Π^*} be the valuation introduced in Lemma 3.7. By Lemma 3.7, it holds that $v_{\Pi^*}(\psi) \neq 0$ for all ψ in Π and $v_{\Pi^*}(\phi) = 0$. Therefore $\Pi \not\models \phi$. □

COROLLARY 3.10. *The system* $\mathbf{ND}_{LP}$ *is sound and complete with respect to* LP.

Proof. Consider the logic $LP\neg$ that is obtained from LP by adding LP's truth table $f_\neg$ for negation to it. Obviously, $LP\neg$ is LP. By Theorems 3.3 and 3.9, $\mathbf{ND}_{LP\neg}$ is sound and complete with respect to $LP\neg$. It is easy to see that the rules $R_\neg(0, 1)$, $R_\neg(i, i)$, and $R_\neg(1, 0)$ are derived rules in $\mathbf{ND}_{LP}$. □

§4. Conclusion. The present investigation of the Logic of Paradox (LP) is only a first step in the development of a full-blown correspondence analysis for many-valued logics.

It is to be expected that similar characterizations can be given of truth table entries of n-ary operators and of truth table entries of many-valued logics that have other sets of truth-values or other sets of designated values than LP. But there is more. Correspondence analysis for many-valued logics raises new theoretical questions and offers simple and powerful techniques to answer them. Let us illustrate this by briefly focusing on a well-known extension of LP: the relevant logic RM_3. This is the logic which we get if we add to LP the following truth table for implication ($\supset$):

$f_\supset$	0	i	1
0	1	1	1
i	0	i	1
1	0	0	1

From Theorem 3.9 it follows that if we add the nine derivation rules that characterize the nine entries in the truth table $f_\supset$ to the basic natural deduction system $\mathbf{ND}_{LP}$, we obtain a natural deduction system which is sound and complete with respect to RM_3. Let us call this natural deduction system $\mathbf{ND}_{RM_3}$.

As a consequence of its completeness with respect to RM_3, all the axioms and derivation rules of the axiomatizations of RM_3 in the literature (Anderson & Belnap, 1975, pp. 469–470; Brady, 1982) are provable in $\mathbf{ND}_{RM_3}$. Which of the nine derivation rules that characterize the truth table $f_\supset$ are necessary and sufficient for which axioms and derivation rules in these axiomatizations of RM_3? With answers to this question, we can systematize the contribution of an individual entry in the truth table $f_\supset$ to the overall properties of implication in RM_3. Let us just list some preliminary results.

Against the background of $\mathbf{ND}_{LP}$ we can show that the rules $R_\supset(i, 0, 0)$ and $R_\supset(1, 0, 0)$ taken together are deductively equivalent to $\phi, \phi \supset \psi \vdash \psi$, that the rules $R_\supset(1, 0, 0)$ and $R_\supset(1, i, 0)$ taken together are deductively equivalent to $\phi \supset \psi, \neg\psi \vdash \neg\phi$, that the rules $R_\supset(0, 0, 1)$, $R_\supset(0, i, 1)$, $R_\supset(0, 1, 1)$ taken together are deductively equivalent to $\neg(\phi \supset \psi) \vdash \phi$, and that the rules $R_\supset(0, 1, 1)$, $R_\supset(i, 1, 1)$, and $R_\supset(1, 1, 1)$ taken together are deductively equivalent to $\neg(\phi \supset \psi) \vdash \neg\psi$. Hence, these four derivation rules characterize eight out of nine entries in the truth table $f_\supset$. Adding these four derivation rules and the derivation rule $R_\supset(i, i, i)$ to the basic system $\mathbf{ND}_{LP}$ therefore yields another natural deduction system that is sound and complete with respect to RM_3.

In summary, correspondence analysis for many-valued logics greatly facilitates proof-theoretic investigations of these logics. It helps us to explore uncharted territory and opens up new perspectives. Where it will all lead us is yet to be seen.

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